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ERGODIC TYPE THEOREMS FOR FINITE VON NEUMANN ALGEBRAS

BY

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ABSTRACT

Ergodic type theorems for automorphisms of finite von Neumann algebras are considered. Neveu decomposition was employed in order to prove stochastical convergence.

Introduction and notations

This paper is devoted to the presentation of some results concerning ergodic type theorems in finite von Neumann algebras. The first results in this field were obtained by Sinai and Anshelevich [14] and Lance [11]. Development of the subject is reflected in the monographs of Jajte [5] and Krengel [10].

The notion of a weakly wandering set (in a commutative context) was introduced by Hajian and Kakutani [7] in order to establish conditions which are equivalent to the existence of finite invariant measures. The non-commutative case was considered by Jajte [6].

In section 1 we consider Neveu decomposition which gives a characterization of the existence of the invariant measure in terms of a weakly wandering operator.

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Section 2 is devoted to a representation of the Krengel stochastic ergodic theorem for the action of an automorphism on finite von Neumann algebra [3].

In section 3 we consider a multiparametric version of the stochastic ergodic theorem. In section 4, the multiparametric superadditive stochastic ergodic theorem is considered.

We use the following notations: M is a von Neumann algebra with finite tracial state τ, M_* is a predual of M, and M^* is the Banach dual space to M; $\mathbb{I}$ denotes the unit of M. For $\rho \in M_*$, the support of ρ will be denoted by $S(\rho)$. Let α be an automorphism of algebra M, and let α_* be an operator acting in M_*, to which α is conjugated. By A^n (A^{*n}) we denote the Česaro average of α (α^*).

1. Neveu decomposition and the weakly wandering operator

Definition 1.1: A positive operator $h \leq \mathbb{I}$ is said to be a **weakly wandering operator**, if

$$||A^n h|| \to 0 \quad \text{when } n \to \infty.$$

The following theorem is valid.

THEOREM 1.1 (see [7], [6], [10]): *Let M and α be as defined above. The following conditions are equivalent:*

(i) *There exists an α_*-invariant normal state ρ on M with support $S(\rho) = E$ such that the support of every α_*-invariant normal state μ is less than or equal to E: $S(\mu) \leq E$.*

(ii) *E is the maximal projection such that for every projector $P \leq E$, $P \in M$,*

$$\inf_n \tau(\alpha^n P) > 0.$$

(iii) *There exists a weakly wandering operator $h_0 \in M_+$ with support $S(h_0) = \mathbb{I} - E$ such that the support of every weakly wandering operator is less than or equal to $\mathbb{I} - E$.*

Proof: (i) $\Rightarrow$ (ii). Let $e \leq E$ and $\rho(e) = \varepsilon > 0$. Since the set $\{\nu\colon \nu \leq \lambda\tau,\ \nu \in M_{*1}^+,\ \lambda = 1, 2 \ldots\}$ is dense in norm in M_{*1}^+, then there exist $\lambda > 1$ and $\nu \in M_{*1}^+$ such that $\nu \leq \lambda\tau$, $||\rho - \nu||_1 < \varepsilon/2$. So

$$\tau(\alpha^n e) \geq \lambda^{-1}\nu(\alpha^n e) \geq \lambda^{-1}(\rho(\alpha^n e) - ||\rho - \nu||_1) > \lambda^{-1} \cdot \varepsilon/2 > 0.$$

Let (i) hold and E_1 be a projection such that not $E \geq E_1$ and for every $P \leq E_1$ the following relation is valid:

$$\inf_{n\geq 0} \tau(\alpha^n P) > 0.$$

Let us consider the sequence $\{A^{*n}\tau\}_{n=1}^{\infty} \subset M^*$. By the Banach–Alaogly theorem M_1^* is compact with respect to $\sigma(M^*, M)$ topology; we denote the limit point of $\{A^{*n}\tau\}_{n=1}^{\infty}$ by ν_0, $\nu_0 \in M_1^*$. From $A^{*n}\tau \geq 0$ it follows that $\nu_0 \geq 0$. Moreover,

$$||A^{*n}\tau - A^{*n}(\alpha^*\tau)|| \leq \frac{2}{n} \cdot ||\tau||,$$

so the limit state ν_0 is α^* invariant. Let $\nu_0 = \nu_{0n} + \nu_{0s}$ be a Takesaki decomposition [15] of state ν_0 on normal and singular components. We show that the normal component is non-zero. Otherwise, for every projector q there exists projector $p \leq q$, $p \neq 0$ with $\nu_{0s}(P) = 0$ [15] or $\inf_{n\geq 0}(A^{*n}\tau)(P) = 0$, but this contradicts the assumption $\inf_{n\geq 0}\tau(\alpha^n P) > 0$. Besides, the normal component is α_* invariant. From $\alpha^* M_* \subset M_*$ it follows that

$$\alpha^*\nu_0 = \alpha^*\nu_{0n} + (\alpha^*\nu_{0s})_n + (\alpha^*\nu_{0s})_s,$$

where $(\alpha^*\nu_{0s})_{n(s)}$ is the normal (singular) component of functional $\alpha^*\nu_{0s}$. Since α is an automorphism, so $(\alpha^*\nu_{0s})_s = \alpha^*\nu_{0s}$ and $\alpha^*\nu_{0n} = \alpha_*\nu_{0n} = \nu_{0n}$. By the choice of projector E_1 we have $S(\nu_{0n}) \geq E_1$ since $\nu_{0n}(P) \geq \inf_{n\geq 0}(A^{*n}\tau)(p') \geq \inf_{n\geq 0}\tau(\alpha^n p') > 0$, where $p' \leq P \leq E_1$, $\nu_{0s}(p') = 0$, $p' \neq 0$, or ν_{0n} is faithful on E_1. This contradicts (i).

(ii) $\Rightarrow$ (i) It is easy to check that the set of all projectors satisfying (ii) is closed with respect to countable supremum. It follows from σ-finiteness of algebra M that there exists a maximal projector E satisfying (ii). Let ν_n be an invariant normal state associated with E. Then $S(\nu_n) \geq E$; however, for $P \leq S(\nu_n)$ condition (ii) is valid. It follows from maximality of E that $S(\nu_n) = E$.

(i), (ii) $\Rightarrow$ (iii) First of all we note that for every projector $P \leq \mathbb{I} - E$ there exists a non-zero projector $q \leq P$ such that $\inf_{n\geq 0}\tau(A_n q) = 0$. Otherwise construction from implication (i) $\Rightarrow$ (ii) gives an invariant state μ with support $S(\mu) \geq P$, which contradicts (i). Let $\tau(q) = \varepsilon > 0$. There exists a sequence of naturals $n_1, n_2 \ldots, n_k, \ldots$ such that $n_1 < n_2 < \cdots < n_k < \cdots$ and

$$\tau(\bigvee_{j=1}^{k-1} \alpha^{n_k - n_j} q) < \varepsilon \cdot 2^{-(k+1)}.$$

The proof is by induction on k.

$k = 1$: From $\inf_{n\geq 0}\tau(A_n q) = 0$ it follows that there exists n_1 with $\tau(\alpha^{n_1}q) < \varepsilon\cdot 2^{-2}$.

Let us assume that $n_1, \ldots, n_{k-1}$ are already defined. Note that

$$\inf \tau(A_n(q + \alpha^{n_{k-1}-n_{k-2}}q + \cdots + \alpha^{n_{k-1}-n_1}q)) = 0.$$

Indeed, $|\tau(A_n q) - \tau(A_n(\alpha^m q))| \leq n^{-1}2m$, hence

$$\inf_{n\geq 0} \tau(A_n(q + \alpha^{n_{k-1}-n_{k-2}}q + \cdots + \alpha^{n_{k-1}-n_1}q)) \leq \inf_{n\geq 0}\tau(A_n q) + 2kn_{k-1}\cdot n^{-1}.$$

Then find n_k in such a way that

$$\tau(\bigvee_{j=1}^{k-1} \alpha^{n_k-n_j}q) \leq \sum_{j=1}^{k-1}\tau(\alpha^{n_k-n_j}q) < \varepsilon\cdot 2^{-(k+1)}.$$

Let

$$q_1 = \bigvee_{k=1}^{\infty}\bigvee_{j=0}^{k-1}\alpha^{n_k-n_j}q \quad\text{and}\quad q_2 = q\wedge q_1^{\perp},$$

where $q_1^{\perp} = \mathbb{I} - q_1$. Let us show that $q_2 \neq 0$. Indeed,

$$\tau(q_2^{\perp}) = \tau(q^{\perp}\vee q_1) = \tau(q^{\perp}\vee(\bigvee_{k=1}^{\infty}\bigvee_{j=0}^{k-1}\alpha^{n_k-n_j}q)) \leq 1 - \varepsilon\cdot 2^{-1} < 1.$$

From $\alpha^{n_k-n_\ell}q \leq q_1$ for $k > \ell$ it follows that $\alpha^{n_k-n_\ell}q\perp q_1^{\perp}$ for $k > \ell$. So $\alpha^{n_k-n_\ell}q_2 = \alpha^{n_k-n_\ell}q \wedge \alpha^{n_k-n_\ell}q_1^{\perp} \leq \alpha^{n_k-n_\ell}q$ and $\alpha^{n_k-n_\ell}q_2\perp q_1^{\perp}$. Hence $\alpha^{n_k-n_\ell}q_2\perp q_2$ for $k > \ell$ and $\alpha^{n_k}q_2\perp\alpha^{n_\ell}q_2$, $k > \ell$. Thus for $P \leq \mathbb{I} - E$ the non-zero projector $q_2 \leq p$ was constructed such that $\alpha^{n_k}a_2\perp\alpha^{n_\ell}a_2$ for some sequence $n_1 \leq n_2 \leq \cdots$ and

$$\Big\|\sum_{j=1}^{\infty}\alpha^{n_k}q_2\Big\|_{\infty} = 1.$$

From the estimate

$$||A^N q_2|| \leq \frac{2n_k\cdot k}{N} + \frac{1}{k}||A^N(q_2 + \alpha^{n_1}q_2 + \cdots + \alpha^{n_k}q_2)|| \leq k\cdot 2^{k-1} + k^{-1}$$

for $N > n_k\cdot 2^k$ there follows convergence to zero when $N\to\infty$, so q_2 is a weakly wandering operator.

Let $q_{2,1} = q_2$; the same construction applied to projector $p_{2,1} = \mathbb{I} - E_{q_2}$ gives projector $q_{2,2}$, and so on. Let the system $\{q_{2,n}\}_{n=1}^{\infty}$ be a system with maximal sum. Then $\bar{q} = \mathbb{I} - E$, otherwise the same construction may be used one more time to get projector $\hat{q} \leq \mathbb{I} - E - \bar{q}$, $\hat{q} \neq 0$, what contradicts maximality of $\bar{q}$. The set of such projectors is at most countable; this follows from σ-finiteness of M. Consider operator $h = \sum_{n=1}^{\infty} 2^{-n} q_{2,n}$, $h \geq 0$, $||h|| \leq 1$, by construction of $S(h) = \mathbb{I} - E$. Fix $\varepsilon > 0$, $\ell = [\log_2 \varepsilon^{-1}] + 2$, and natural N_j with $||A_n q_{2,j}|| < \varepsilon \cdot 2^{-j-1}$ when $n \geq N_j$, $M = \max_{i \leq j \leq \ell} N_j$. For $N > M$ the following inequality holds:

$$||A_N h||_\infty \leq \sum_{j=1}^{\ell} ||A_N q_{2,j}||_\infty + 2^{-\ell} \Big\| A_N \Big(\sum_{i=1}^{\infty} 2^{-i} \cdot q_{2,\ell+i} \Big) \Big\|_\infty \leq \varepsilon,$$

or h is a weakly wandering operator.

(iii) $\Rightarrow$ (i) σ-finiteness of M implies that there exists a weakly wandering operator h with the property $S(h) \geq S(h_1)$ for every weakly wandering operator h_1. Let μ be an α_*-invariant normal state with maximal possible support. Then $\mu(h) = 0$ or $S(\mu) \perp S(h)$. Let $p = \mathbb{I} - S(\mu) - S(h)$. Maximality of $S(\mu)$ and implication (i), (ii) $\Rightarrow$ (iii) imply that there exists h_1 with $S(h_1) = \mathbb{I} - S(\mu)$ or $p = 0$. ∎

It follows immediately from the theorem that

COROLLARY 1.1 (Neveu decomposition): *Let α be an automorphism of finite von Neumann algebra M with finite tracial state τ. Then there exist projectors E_1 and E_2, $E_1 + E_2 = \mathbb{I}$ such that*

(i) *There exists an α_*-invariant normal state ρ with support $S(\rho) = E_1$.*

(ii) *There exists a weakly wandering operator $h \in M$ with $S(h) = E_2$.*

2. Stochastic ergodic theorem

The space M_* of normal functionals on von Neumann algebra M with finite trace τ is naturally identified with the space $L_1(M, \tau)$ of measurable operators, each affiliated to M and integrable with modulus. Action α' is defined as an operator conjugated to α with respect to duality:

$$\tau(\alpha' X \cdot y) = \tau(X \cdot \alpha y), \quad X \in L_1(M, \tau), \quad y \in M.$$

Definition 2.1: A sequence $\{X_n\}$ of measurable operators is said to converge stochastically to operator X_0, if for every $\varepsilon > 0$

$$\tau(\{|X_n - X_0| > \varepsilon\}) \to 0.$$

THEOREM 2.1 (Stochastic ergodic theorem; see [10]): *Let α be an automorphism of von Neumann algebra M with tracial state τ. Then for $X \in L_1(M,\tau)$ Česaro averages $A'^n X$ converge stochastically to $\tilde{X} \in L_1(M.\tau)$. The limit $\tilde{X}$ is α'-invariant and $E_2\tilde{X}E_2 = 0$ (where E_2 is a projector from Neveu decomposition).*

Proof: By Theorem 1.1 there exist projectors E_1 and E_2 such that $E_1 + E_2 = \mathbb{I}$, $S(\rho) = E_1$, $S(h) = E_2$, where ρ is an invariant state and h is a weakly wandering operator. Projector E_1 is α invariant. Indeed, $1 = \rho(E_1) = \rho(\alpha E_1)$, so $\alpha E_1 \geq E_1$. Moreover, $\alpha E_1 - E_1$ is a projector and let $E_2 = \alpha^{-1}(\alpha E_1 - E_1)$. Then $E_2 = E_1 - \alpha^{-1}E_1 \leq E_1$, but $\rho(E_2) = 0$; this contradicts that ρ is faithful on E_1. Let $\nu \in M_*$ be a positive normal functional on M, C be associated with μ self-adjoint positive operator, and $C \in L_1(M,\tau)$. Let $\mu(x) = \nu(E_1 x E_1) = \tau(Yx)$ for all $x \in M$. Then $Y = E_1 C E_1$ and $E_1 A_n(\alpha', C)E_1 = A_n(\alpha', E_1 C E_1) = A_n(\alpha', Y)$. Let $B = E_2 C E_2$. Let $\alpha_1 = \alpha|_{M_{E_1}}$, $\alpha_2 = \alpha|_{M_{E_2}}$, $M|_{E_i} = E_i M E_i$, $i = 1,2$. Then $E_2 A_n(\alpha', C)E_2 = A_n(\alpha_2', B)$. Let us show that $A_i'^n X_i$ converge stochastically, where $i = 1,2$, $X_i \in L_1(M_{E_i},\tau)$, and $A_i'^n X_i = n^{-1}\sum_{j=0}^{n-1} \alpha_i'^j X_i$.

LEMMA: *Let E and F be projectors from M, λ_1 and λ_2 be positive numbers, $\lambda_1 < \lambda_2$, and $A \geq 0$, $A \in M$. Then $\lambda_2 F \leq \lambda_1 E + (\mathbb{I} - E)A(\mathbb{I} - E)$ implies that $\tau(F) \leq 1 - \tau(E)$.*

Proof: Let $\tau(F) + \tau(E) > 1$. Then $\tau(F \wedge E) = \tau(E) + \tau(F) - \tau(E \vee F) \geq \tau(E) + \tau(F) - 1 > 0$. Multiplying the inequality in the condition on both sides by $F \wedge E$, one gets $\lambda_2 F \wedge E \leq \lambda_1 \cdot F \wedge E$, and $\lambda_2 \leq \lambda_1$. Contradiction.

Fix $\varepsilon > 0$ and a sequence $\{\delta_i\}$ of positive numbers $\delta_i \downarrow 0$.

We prove convergence in algebra $M|_{E_2}$. Let $B \geq 0$, $B \in L_1(M|_{E_2},\tau)$, $G_i = \{h_0 \leq \lambda_i\}$, $0 < \lambda_i < 1$, such that $\tau(E_2 - G_i) < \delta_i/2$. Inequality $G_i = \lambda_i^{-1}h_0$ implies that G_i is a weakly wandering operator, and for n greater than some number m_i inequality $||A^n(\alpha_2', G_i)||_\infty < \varepsilon \cdot \delta/10$ is valid. One can assume here that sequence $\{m_i\}$ increases. Let $G_{i,n} = \{G_i A^n(\alpha_2', B)G_i \geq \varepsilon/10\}$. Then $\varepsilon/10 \cdot G_{i,n} \leq G_i A^n(\alpha_2', B)G_i$ and since $\tau(G_i A^n(\alpha_2', B)G_i) = \tau(A^n(\alpha_2, G_i)B) \leq \varepsilon \cdot \delta_i \cdot \tau(B)/10$ then $\tau(G_{i,n}) \leq \delta_i/2$. Let $q_{i,n} = G_i - G_{i,n}$. For $n \geq m_i$ one has $\tau(q_{i,n}) \geq \tau(E_2) - \delta_i/2$ and $q_{i,n}A^n(\alpha_2', B)q_{i,n} \in M|_{E_2}$; $||q_{i,n}A^n(\alpha_2', B)q_{i,n}||_\infty \leq \varepsilon/10$ and tends to 0 when $n \to \infty$.

In order to consider convergence in algebra $M|_{E_1}$ we prove the following variant of the individual ergodic theorem.

THEOREM 2.2 (see [17], [5], [4]): *Let M be a von Neumann algebra with finite normal trace τ, $\tau(\mathbb{I}) = 1$. Let α be an automorphism of M, ρ a normal faithful state on M, $\rho \circ \alpha = \rho$. Then for every $\mu \in M_*$ there exists an α_*-invariant normal functional $\bar{\mu}$ such that for every $\varepsilon > 0$ there exists projector $E \in M$, $\tau(E) > 1 - \varepsilon$ and*

$$\sup_{\substack{x \in EM_+E \\ x \neq 0}} |(A_*^n \mu - \bar{\mu})(x)/\tau(x)| \to 0 \quad \text{when } n \to \infty.$$

Let $(H_\rho, \pi_\rho, \mathfrak{M})$ be a representation of algebra M constructed by a faithful normal state ρ. Then $\mathfrak{M}$ is a von Neumann algebra isomorphic to M. Let $\hat{\alpha}$ be an image of automorphism α and $\hat{\alpha}'$ be an associated transformation on $\mathfrak{M}'$:

$$(\hat{\alpha}X \cdot Y\Omega, \Omega) = (X \cdot \hat{\alpha}' Y\Omega, \Omega), \quad X \in \mathfrak{M}, \quad Y \in \mathfrak{M}, \tag{1}$$

where Ω is a bycyclic vector with $(X\Omega, \Omega) = \rho(X)$, $X \in \mathfrak{M}$.

The following theorem is a variant of the maximal Hopf Lemma. Note that such a variant can be used for the proof of individual ergodic theorems in JBW^* algebras.

THEOREM 2.3 (see [4], [16]): *Let $\mu \in \mathfrak{M}_*$ be a hermitian functional and $\varepsilon > 0$, $||\mu|| \cdot \varepsilon^{-1} < 1$. Then for fixed N there exists projector $E \in \mathfrak{M}$, $\rho(E) > 1 - ||\mu|| \varepsilon^{-1}$ such that*

$$\sup_{\substack{x \in E\mathfrak{M}_+E \\ x \neq 0}} |A^n(\hat{\alpha}_*, \mu)(x)/\rho(x)| < \varepsilon, \quad n = 1, 2, \ldots, N.$$

Proof: Denote by $\mathfrak{M}$ von Neumann algebra $\prod_{j=1}^{2N} \mathfrak{M}_j$, where $\mathfrak{M}_j = \mathfrak{M}$. Let

$$L = \{(B_{n,i})_{n=1,\ldots,N;\ i=1,2} \colon 0 \leq B_{n,i} \quad \text{and} \quad \sum_{n=1}^{N} (B_{n,1} + B_{n,2}) < \mathbb{I}\}$$

and define a weakly continuous function on L,

$$G((B_{n,i})) = \sum_{n=1}^{N} n[A^n(\hat{\alpha}_*, \mu')(B_{n,1}) - \rho(B_{n,1}) - (A^n(\hat{\alpha}_*, \mu')(B_{n,2}) - \rho(B_{n,2}))],$$

where $\mu' = \varepsilon^{-1}\mu$.

Weak compactness of L implies that there exists $(\bar{B}_{n,i}) \in L$ such that

$$G((\bar{B}_{n,i})) \geq G((B_{n,i})) \quad \text{for every } (B_{n,i}) \in L.$$

Let $R = \mathbb{I} - \sum_{n=1}^{N}(B_{n,1} + B_{n,2})$, $R' \geq 0$, $R' \leq R$, $(\hat{B}_{n,i}) = \bar{B}_{n,i}$ if $(n,i) \neq (m,j)$ and $\hat{B}_{m,j} = \bar{B}_{m,j} + R'$; then $(\hat{B}_{n,i}) \in L$ and $G((\hat{B}_{n,i})) \leq G((\bar{B}_{n,i}))$.

For $j = 1$ it implies $A^m(\hat{\alpha}_*, \mu')(R') \leq \rho(R')$, for $j = 2$ it implies $A^m(\hat{\alpha}_*, \mu')(R') \geq -\rho(R')$ or $|A^m(\hat{\alpha}_*, \mu)(R')| \leq \rho(R') \cdot \varepsilon$. Let $E = S(R)$ and $0 \leq x \leq \lambda R$. For $R' = \lambda^{-1}x$ it implies that $|A^m(\hat{\alpha}_*, \mu)(x)| \leq \rho(x) \cdot \varepsilon$. The set $\{0 \leq x \leq \mathbb{I},\ x \leq \lambda R$ for some $\lambda > 0\}$ is weakly dense in $\{0 \leq y \leq E\}$, so the estimate above is true for all $0 \leq x \leq E$. It follows from comparing $G((\hat{\alpha}\bar{B}_{n,i}))$ and $G((\bar{B}_{n,i}))$ that $\rho(E) \geq 1 - \varepsilon^{-1}||\mu||$. ∎

By the theorem of Kovacs and Szücs [9], the sequence $A_*^n(\hat{\alpha}_*, \mu)$ converges in the norm in space $\mathfrak{M}_*$ to functional $\bar{\mu}$. Let us show that there is a dense subset $\mathcal{L}$ in the space of normal hermitian functionals $\mathfrak{M}_{*h}$ for which

$$\sup_{\substack{x \in \mathfrak{m}_+ \\ x \neq 0}} |(A^n(\hat{\alpha}_*, \mu) - \bar{\mu})(x)/\rho(x)| \to 0 \quad \text{when } n \to \infty. \tag{2}$$

Let $U = \{\mu \in \mathfrak{M}_{*+}, ||\mu|| \leq 1, \mu \leq \lambda\rho$ for some $\lambda > 0\}$. For $\mu \in U$ there exists $y \in \mathfrak{M}'_+$, $y \leq \mathbb{I}$ such that $\mu(x) = (yx\Omega, \Omega)$ [3]. So

$$A^n(\hat{\alpha}_*, \mu(x)) = \mu(A^n x) = (y \cdot A^n x\Omega, \Omega) = (A^n(\hat{\alpha}', y)x\Omega, \Omega).$$

From $\hat{\alpha}_*$-*invariance* of ρ it follows that $\bar{\mu} \leq \lambda\rho$ and there exists $\bar{y} \in \mathfrak{M}'$ with $\bar{\mu}(x) = (\bar{y}x\Omega, \Omega)$; $x \in \mathfrak{M}$. Let $y_n = y - A^n(\hat{\alpha}', y) + \bar{y}$ and $\mu_n(x) = (y_n x\Omega, \Omega)$; then for $x \in \mathfrak{M}_+$

$$|(A^k(\hat{\alpha}_*, \mu_n) - \bar{\mu})(x)| = |(A^k(\hat{\alpha}', (y - y_n))x\Omega, \Omega)| \leq \frac{2n||y||_\infty}{k} \cdot (x\Omega, \Omega)$$

since

$$||A^k(\hat{\alpha}', y - y_n)||_\infty \leq \frac{2n||y||_\infty}{k}.$$

Moreover,

$$||\mu_n - \mu||_1 = \sup_{\substack{x \in \mathfrak{m} \\ ||x||_\infty \leq 1}} |(\bar{y} - A^n(\hat{\alpha}', y)) \cdot x\Omega, \Omega)| = ||A^n(\hat{\alpha}_*, \mu) - \bar{\mu}||_1 \to 0$$

when $n \to \infty$. So $\mathcal{L} = \{\mu_n - \nu_n : \mu \leq \lambda\rho,\ \nu \leq \lambda\rho,\ \lambda > 0,\ \mu, \nu \in \mathfrak{M}_{*+}\}$ is dense in norm in the set of all normal hermitian functionals.

Proof of Theorem 2.2: There exists a self-adjoint positive operator B affiliated with algebra M and τ-integrable with modulus such that $\rho(x) = \tau(Bx)$, $x \in$

M. We will identify normal functionals on M and normal functionals on $\mathfrak{M}$. Let us show that there exists function $\gamma(\alpha)$ such that for every projector $E \in M$ inequality $\rho(E) \leq \alpha$ implies $\tau(E) \leq \gamma(\alpha)$ and $\gamma(\alpha) \to 0$ when $\alpha \to 0$. Consider unit ball M_1 with metrics $\rho_1(x,y) = \rho((x-y)^*(x-y))$ and $\tau_1(x,y) = \tau((x-y)^*(x-y))$. By the theorem [3], faithfulness of ρ and τ on M implies that topologies generated by metrics ρ_1 and τ_1 coincide with strong topology and the metrics are equivalent. Existence of function $\gamma(\alpha)$ is exactly equivalence of distance from zero to projector E.

Fix $\varepsilon > 0$. Let $\{\mu_j\}$ be a sequence of normal functionals satisfying (2) and

$$||\mu - \mu_j|| \leq 4^{-j} \cdot \gamma_j, \quad j = 1, 2, \ldots; \; N_j \tag{3}$$

be naturals such that

$$\sup_{\substack{x \in M_+ \\ x \neq 0}} |(A^n(\alpha_*, \mu_{j+1}) - \bar{\mu}_{j+1}(x))|/\rho(x) \leq 4^{-j-2}\gamma_j \quad \text{for } n \geq N_j,$$

where $\bar{\mu}_j$ is a $||\cdot||_1$-$\lim A^n(\alpha_*, \mu_j)$, existence of the limit follows from the Kovacs–Szücs theorem, $\gamma_j = \max(\gamma^{-1}(\varepsilon \cdot 4^{-j}), 1)$, and $\gamma^{-1}(\alpha)$ is the inverse function of γ.

By applying Theorem 2.3 to the functional $\mu - \mu_j$ with $\varepsilon' = 4^{-j}\gamma_j$ and $N = N_j$, there exists a projector E_j with $\rho(E_j) > 1 - \gamma_j$ such that

$$\sup_{\substack{x \in E_j M_+ E_j \\ x \neq 0}} |A^n(\alpha_*, \mu - \mu_j)(x)|/\rho(x) < 4^{-j}.$$

Moreover, going to the limit in norm of M in (3) implies $||\bar{\mu} - \bar{\mu}_j|| \leq 4^{-j}\gamma$; so there exists F_j with $\rho(F_j) > 1 - \gamma_j$ and

$$\sup_{\substack{x \in F_j M_+ F_j \\ x \neq 0}} |(\bar{\mu} - \bar{\mu}_j)(x)|/\rho(x) \leq 4^{-j-1}.$$

Let $E'_j = E_j \wedge F_j$. Then $\tau(E'_j) \geq 1 - \tau((\mathbb{I} - E_j) \vee (\mathbb{I} - F_j)) \geq 1 - 2 \cdot \varepsilon \cdot 4^{-j}$, since $\tau(\mathbb{I} - E_j) \leq \varepsilon \cdot 4^{-j}$. Let $E' = \bigwedge_j E'_j$. Then $\tau(\mathbb{I} - E') \leq \sum_{j=1}^{\infty} \tau(\mathbb{I} - E'_j) \leq 2\varepsilon/3$. Let $F' = \{\lambda^{-1} \leq B \leq \lambda\}$, $\lambda > 1$, λ such that $\tau(F') > 1 - \varepsilon/6$. Then for $E = E' \wedge F'$, $\tau(E) > 1 - \varepsilon$. Fix $\delta > 0$, $\delta < 1$ and $N = |\log_{1/4} \delta \cdot \lambda^{-1}| + 1$. Let j be such that $N_j \leq N \leq N_{j+1}$, $n \geq N_{j+1}$. Then

$$\sup_{\substack{x \in E M_+ E \\ x \neq 0}} |(A^n(\alpha_*, \mu) - \bar{\mu})(x)|/\tau(x)$$

$$\leq \sup_{\substack{x\in EM_+E\\ x\neq 0}} |(A^n(\alpha_*,\mu-\mu_j)-(\bar\mu-\bar\mu_j))(x)|/\tau(x)$$

$$+ \sup_{\substack{x\in EM_+E\\ x\neq 0}} |A^n(\alpha_*,\mu_j)-\bar\mu_j)(x)|/\tau(x)$$

$$(4)\qquad \leq \sup_{\substack{x\in EM_+E\\ x\neq 0}} |(A^n(\alpha_*,\mu-\mu_j)-(\bar\mu-\bar\mu_j))(x)|/\rho(x)\cdot \sup_{\substack{x\in EM_+E\\ x\neq 0}} \rho(x)/\tau(x)$$

$$+ \sup_{\substack{x\in EM_+E\\ x\neq 0}} |(A^n(\alpha_*,\mu_j)-\bar\mu_j)(x)|/\rho(x) \sup_{\substack{x\in EM_+E\\ x\neq 0}} \rho(x)/\tau(x)$$

$$\leq 4^{-j}\cdot \sup_{\substack{x\in EM_+E\\ x\neq 0}} \rho(x)/\tau(x).$$

Taking into consideration that $0\leq x\leq E$ we have $\rho(x)/\tau(x)=\tau(B\cdot F'x)/\tau(x)\leq \tau(\lambda F'x)/\tau(x)\leq\lambda$. So (3) does not exceed $4^{-j}\lambda\leq\delta$ by the choice of j. Theorem 2.2 is proved.

Continuation of proof of Theorem 2.1: By Theorem 2.2 there exist projectors P_j and naturals N_j such that $\tau(E_1-P_j)<\delta_j$ and

$$\sup_{\substack{x\in P_jM_+P_j\\ x\neq 0}} |(A^n(\alpha_*,\mu)-\bar\mu)(x)|/\tau(x)<\varepsilon/10, \qquad \text{when } n>N_j, \quad j=1,2,\ldots.$$

Let $Y,\bar Y\in L_1(M|_{E_1},\tau)$ be self-adjoint operators such that $\tau(Yx)=\mu(x)$ and $\tau(\bar Yx)=\bar\mu(x), x\in M|_{E_1}$ (remember that $\bar\mu$ is a $||\ ||_1-\lim A^n(\alpha_*,\mu)$. Then

$$\sup_{\substack{x\in P_jM_+P_j\\ x\neq 0}} |(A_n(\alpha_{1*},\mu-\bar\mu)(x)|/\tau(x)$$

$$= \sup_{\substack{x\in P_jM_+P_j\\ x\neq 0}} |(\mu-\bar\mu)(A_n(\alpha_1,x))|/\tau(x)$$

$$= \sup_{\substack{x\in P_jM_+P_j\\ x\neq 0}} |\tau((Y-\bar Y)(A_n(\alpha_1,x)))|/\tau(x)$$

$$= \sup_{\substack{x\in P_jM_+P_j\\ x\neq 0}} |\tau((A_n(\alpha_1',Y)-\bar Y)x)|/\tau(x)$$

$$= \sup_{\substack{z^*\in P_jM\\ z^*\neq 0, x=z^*z}} |((A^n(\alpha_1',Y)-\bar Y)z^*\xi_0,z^*\xi_0)|/(z^*\xi_0,z^*\xi_0)$$

$$= \sup_{\substack{\eta\in P_iH\\ ||\eta||\leq 1}} |(A_n(\alpha_1',Y)-\bar Y)\eta,\eta)) = ||P_j(A_n(\alpha_1',Y)-\bar Y))P_j||_\infty.$$

Let $A_n(\alpha', Y) - \bar{Y} = B_{n,+} - B_{n,-}$, $S(B_{n,+}) \cdot S(B_{n,-}) = 0$.

Let $F_{n,j} = E_1 - S(B_{n,+}) - S(B_{n,-}) + S(B_{n,+}) \wedge P_j + S(B_{n,-}) \wedge P_j$. Since $\tau(S(B_{,\pm}) \wedge P_j) \leq \tau(S(B_{n,\pm})) - \delta_j$ then $\tau(F_{n,j}) \geq 1 - 2\delta_j$. Moreover,

$$\begin{aligned}
&||F_{n,j}(A_n(\alpha_1', Y) - \bar{Y})F_{n,j}||_\infty^2 \\
&= ||F_{n,j}(A_n(\alpha_1', Y) - \bar{Y})[S(B_{n,+}) \wedge P_j + S(B_{n,-}) \wedge P_j \\
&\quad + (\mathbb{I} - S(B_{n,+}) - S(B_{n,-}))(A_n(\alpha_1', Y) - \bar{y})F_{n,j}||_\infty \\
&\geq \max\{||F_{n,j}(A_n(\alpha_1', Y) - \bar{Y})S(B_{n,+}) \wedge P_j||_\infty^2; \\
&\qquad ||F_{n,j}(A_n(\alpha', Y) - \bar{Y})S(B_{n,-}) \wedge P_j||_\infty^2; \\
&\qquad ||F_{n,j}(A_n(\alpha_1', Y) - \bar{Y})(\mathbb{I} - S(B_{n,+}) - S(B_{n,-})||_\infty^2\} \\
&= \max\{||F_{n,j}B_{n,+}F_{n,j}||_\infty^2, ||F_{n,j}B_{n,-}F_{n,j}||_\infty^2\},
\end{aligned}$$

where inequality is valid because $||C + D + E||_\infty \geq \max\{||C||_\infty, ||D||_\infty, ||E||_\infty\}$ for $C, D, E \geq 0$ and the next equality is valid because

$$\begin{aligned}
||F_{n,j}(A_n(\alpha_1', Y) - \bar{Y})S(B_{n,\pm}) \wedge P_j||_\infty^2 &= ||F_{n,j}(B_{n,+} - B_{n,-})S(B_{n,\pm}) \wedge P_j||_2^2 \\
&= ||F_{n,j}B_{n,\pm}S(B_{n,\pm}) \wedge P_j||_\infty^2 \\
&= ||F_{n,j}B_{n,\pm}F_{n,j}||_\infty^2.
\end{aligned}$$

Let $E_{i,n} = q_{i,n} \wedge Q_{i,n} + F_{n,i} \wedge Q_{i,n}$, where projectors $Q_{i,n}$ will be chosen later. Then

$$\sup_{\substack{\xi \in H \\ ||\xi|| < 1}} |(E_{i,n}(A^n(\alpha', C) - \bar{Y})E_{i,n}\xi, \xi)|$$

$$\begin{aligned}
&\leq \sup_{\substack{\xi \in H \\ ||\xi|| < 1}} |(q_{i,n}E_2(A^n(\alpha', C) - \bar{Y})E_2 q_{i,n}\xi, \xi)| \\
&+ \sup_{\substack{\xi \in H \\ ||\xi|| < 1}} |(F_{n,i}E_1(A^n(\alpha', C) - \bar{Y})E_1 F_{n,i}\xi, \xi)| \qquad (5) \\
&+ \sup_{\substack{\xi \in H \\ ||\xi|| < 1}} |((q_{i,n} \wedge Q_{i,n}A^n(\alpha', C)F_{n,i} + F_{n,i}A^n(\alpha', C)q_{i,n} \wedge Q_{i,n})\xi, \xi)|.
\end{aligned}$$

The first term of inequality (5) does not exceed

$$\sup_{\substack{\xi \in H \\ ||\xi|| \leq 1}} |(q_{i,n}A^n(\alpha_2', B)q_{i,n}\xi, \xi)| < \varepsilon/10$$

and tends to 0 when $n \to \infty$. The second term of (5) does not exceed

$$\sup_{\substack{\xi \in H \\ \|\xi\| \leq 1}} |(F_{n,i}(A^n(\alpha_1', Y) - \bar{Y})F_{n,i}\xi, \xi)| < \varepsilon/10$$

and tends to 0 when $n \to \infty$. Let $Q_{i,n} = \{A^n(\alpha', C) \leq \lambda_{i,n}\}$, where $\lambda_{i,n} \geq 0$. Since $\tau(Q_{i,n}^{\perp}) \leq \lambda_{i,n}^{-1} \cdot \tau(C)$, the choice $\lambda_{i,n} > \tau(C)/\delta_i$ implies $\tau(Q_{i,n}) \geq 1 - \delta_i$. Then

$$\begin{aligned} &g_{i,n} \wedge Q_{i,n} A^n(\alpha', C) F_{n,i} \wedge Q_{i,n} A^n(\alpha', C) g_{i,n} \wedge Q_{i,n} \\ &\leq g_{i,n} \wedge Q_{i,n} A^n(\alpha', C) Q_{i,n} A^n(\alpha', C) g_{i,n} \wedge Q_{i,n} \\ &\leq \lambda_{i,n} g_{i,n} \wedge Q_{i,n} A^n(\alpha', C) g_{i,n} \wedge Q_{i,n} \\ &\leq \lambda_{i,n} g_{i,n} E_2 A^n(\alpha', C) E_2 g_{i,n} = \lambda_{i,n} g_{i,n} A^n(\alpha_2', B) g_{i,n}, \end{aligned}$$

so

$$\|g_{i,n} \wedge Q_{i,n} A^n(\alpha', C) F_{i,n} \wedge Q_{i,n}\|_\infty \leq (\lambda_{i,n} a_{n,i})^{1/2}.$$

Thus the left part of (5) does not exceed $a_{n,i} + b_{n,i} + 2(\lambda_{i,n} a_{n,i})^{1/2}$. Fix i. The estimate on $\lambda_{i,n}$ depends only on i while $b_{n,i}$, and $a_{n,i}$ tend to zero when $n \to \infty$, so there exist naturals K_i such that, for $n > K_i$, the estimate $a_{n,i} + b_{n,i} + s(\lambda_{i,n} a_{n,i})^{1/2} < \varepsilon$ is valid. It is possible to assume that K_i increase. For $n \in [K_i, K_{i+1})$,

$$\|E_{i,n}(A^n(\alpha', C) - \bar{Y})E_{i,n}\|_\infty < \varepsilon$$

and

$$\tau(E_{i,n}) = \tau(q_{i,n} \wedge Q_{i,n} + F_{n,i} \wedge Q_{i,n}) > 1 - 4\delta_i$$

hold.

Let $P_n' = \{|A_n(\alpha', C) - \bar{Y}| \geq 5\varepsilon\}$. Then

$$\begin{aligned} 5\varepsilon \cdot P_n' &\leq |A_n(\alpha', C) - \bar{Y}| \\ &\leq 2 \cdot E_{i,n} \cdot |A_n(\alpha', C) - \bar{Y}| E_{i,n} + 2(\mathbb{I} - E_{i,n})|A_n(\alpha', C) - \bar{Y}|(\mathbb{I} - E_{i,n}) \\ &\leq 4\varepsilon E_{i,n} + 2(\mathbb{I} - E_{i,n})|A_n(\alpha', C) - \bar{Y}|(\mathbb{I} - E_{i,n}). \end{aligned}$$

By the lemma $\tau(P_n') \leq 1 - \tau(E_{i,n}) \leq 4\delta_i$. ∎

3. Case of d commuting automorphisms

Let us consider the case of d-commuting automorphisms. Let $d \geq 1$ be a natural number and $\mathbb{V} = \{0, 1, 2, \ldots\}^d$ be an additive semigroup of d-dimensional vectors with natural coordinates. For $u = (u_i), v = (v_i) \in \mathbb{V}$, relation $u \geq v$ ($u > v$) means $u_i \geq v_i$ ($u_i > v_i$) for $i = 1, \ldots, d$. By $[u, v[$ we denote the set $\{\omega \in \mathbb{V} : u \leq \omega < v\}$. For a finite set B let card (B) or $|B|$ mean the number of elements of B. For $n = (n_1, \ldots, n_d) \in \mathbb{V}$ let $\pi(n) = \prod_{\nu=1}^d n_\nu = |[0, n[|$. For $n \in \mathbb{V}$ and operators $\beta_1, \beta_2, \ldots, \beta_d$ let $\beta_n = \beta_1^{n_1} \beta_2^{n_2} \ldots \beta_d^{n_d}$; $S_n = \sum_{u \in [0,n[} \beta_u$; $A_n = \pi(n)^{-1} S_n$; expression $n \to \infty$ means that n_ν tends to infinity independently for $\nu = 1, 2, \ldots, d$. Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be automorphisms of algebra M. Operator $h \in M_+^1$ is called weakly wandering if $||A_n h||_\infty \to 0$ when $n \to \infty$. A multisequence $\{X_n\}_{n \in \mathbb{V}}$ of measurable operators affiliated with M is said to converge stochastically to operator X_0, if for every $\varepsilon > 0$ $\tau(\{|X_n - X_0| > \varepsilon\}) \to 0$ holds when multiindex $n \to \infty$.

THEOREM 3.1 (see [10]): *Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be commuting automorphisms on von Neumann algebra M with faithful normal finite trace τ. The following conditions are equivalent:*

(i) *There exists an $\alpha_{*,i}$-invariant normal state ρ on M with support E such that the support of every $\alpha_{*,i}$-invariant normal state does not exceed E ($i = 1, 2, \ldots, d$).*

(ii) *There exists a weakly wandering operator $h_0 \in M_+$ with support $\mathbb{I} - E$ such that the support of every weakly wandering operator does not exceed $\mathbb{I} - E$.*

Moreover $E = \bigwedge_{i=1}^d E_i$; $\mathbb{I} - E = \bigvee_{i=1}^d (\mathbb{I} - E_i)$, where E_i is the "maximal" support of the invariant normal state of the automorphism α_i, $i = 1, 2, \ldots, d$.

Proof: Let $E_1 \in M$ be the "maximal" support of an α_1-invariant state ρ. Then algebra $M|_{E_1}$ is α_1 invariant, $i = 1, 2, \ldots, d$. Let us show first that the support of the normal component of every $\sigma(M^*, M)$ limit point of the set $\{A^n \mu\}_{n=1}^\infty$ where $\mu \in M_*^+$; $\mu(\mathbb{I}) = \mathbb{I}$, $S(\mu) = \mathbb{I}$ is equal to E_1. Indeed, let $e \leq E_1$ be a non zero projector from M, $\rho(e) = a > 0$. Let $\lambda > 0$ and let $\nu \leq \lambda\mu$ be a normal functional with $||\rho - \nu|| \leq a/2$. Let μ_0 be a $\sigma(M^*, M)$ limit point of $\{A^{*n}\mu\}_{n=1}^\infty$ and $\{n_\gamma\}$ be a set such that $A^{*n_\gamma}\mu \to \mu_0$, $A^{*n_\gamma}\nu \to \nu_0$, where $\nu_0 \in M_+^*$, and let $\nu_0 = \nu_{0n} + \nu_{0s}$ be a Takesaki decomposition of functional ν_0 on normal and singular components. Then

$$||\rho - \nu_0||_1 \le \limsup_{n_\gamma} ||A^{*n_\gamma}(\rho - \nu)|| \le ||\rho - \nu||_1 < a/2.$$

There exists [15] a projector such that $e' \le e$, $e' \in M$, $\rho(e') > 3/4a$, $\nu_{0s}(e') = 0$. Then $\nu_{0n}(e') = \nu_0(e') \ge \rho(e') - ||\rho - \nu_0|| > a/4$ or $\mu_{0n}(e') \ge \lambda^{-1}\nu_{0n}(e') > \lambda^{-1}a/4$ since relation $\mu_0 \ge \lambda^{-1}\nu_0$ implies $\mu_{0n} \ge \lambda^{-1}\nu_{0n}$ [15]. Arbitrariness of projector P implies that μ_{0n} is faithful on E_1ME_1 or $S(\mu_{0n}) \ge E_1$. It follows from maximality of E_1 that $S(\mu_{0n}) = E_1$.

For $\bar{E} \in M$, we have

$$(\alpha_2^*\mu_0)(\bar{E}) = \lim_{n_\gamma}(\alpha_2^* A^{*n_\gamma}\mu)(\bar{E}) = \lim_{n_\gamma}(A^{*n_\gamma}\alpha_2^*\mu)(\bar{E}). \tag{6}$$

Functional $\alpha_2^*\mu$ is faithful and normal because μ is faithful and normal. So the normal component of $\alpha_2^*\mu_0$ has support E_1. Then $\alpha_2(\mathbb{I} - E_1) = \mathbb{I} - E_1$ because $(\alpha_2^*\mu_{0n})(\mathbb{I} - E_1) = 0$, so $0 = E_1\alpha_2(\mathbb{I} - E_1)E_1$; $||E_1\alpha_2(\mathbb{I} - E_1)E_1||_\infty = ||\alpha_2(\mathbb{I} - E_1)E_1E_1||_\infty = 0$ and $\alpha_2(\mathbb{I} - E_1) = (\mathbb{I} - E_1)\alpha_2(\mathbb{I} - E_1)(\mathbb{I} - E_1) = \mathbb{I} - E_1$. Thus $M|_{E_1}$ is α_i invariant for $i = 1, 2, \dots, d$. Relation (6) implies also that functional $\alpha_{*2}\mu_{0n}$ is α_{*1} invariant. Let us show that there exists an α_{*i}-invariant normal state with support not less than $\bigwedge_{i=1}^d E_i$. Let $e \le \bigwedge_{i=1}^d E_i$, $e \ne 0$. Then

$$\inf_n \tau(E_1(E_1\alpha_2E_1)^n(e)E_1) = \inf_n \tau(\alpha_2^n e) > 0,$$

and an analogous relation holds for every $e' \le e$, $e' \ne 0$. By Theorem 1(ii) there exists an $(E_1\alpha_2E_1)_*$-invariant normal state on algebra E_1ME_1 with support $\bar{E}_2$ such that $\bar{E}_2 \ge e$ and $\bar{E}_2$ is "maximal" in the sense of point (i) of the theorem. By what was proved above automorphisms $E_1\alpha_iE_1$ $(i = 1, 2, \dots, d)$ leave $\bar{E}_2$ invariant, so all considerations can be extended on algebra $\bar{E}_2M\bar{E}_2$. Relation (6) also holds if we change E_1 to $\bar{E}_2$ and α_2 to α_3. Repeating the procedure one gets an $\bar{E}_{d-1}\alpha_d\bar{E}_{d-1}$-invariant normal state with support $\bar{E}_d \ge e$; projector $\bar{E}_d$ is α_i invariant $(i = 1, 2, \dots, d)$. Arbitrariness of $e \le \bigwedge_{i=1}^{d} E_i$ implies the proposition.

Let h_i be a weakly α_i-wandering operator with "maximal" support $(i = 1, 2, \dots, d)$. Then for $n = (n_1, n_2, \dots, n_d)$

$$||A_nh_i||_\infty = ||A_{\alpha_1}^{n_1}A_{\alpha_2}^{n_2}\dots A_{\alpha_d}^{n_d}A_{\alpha_i}^{n_i}h_i||_\infty \le ||A_{\alpha_i}^{n_i}h_i||_\infty \to 0 \quad \text{when } n \to \infty.$$

Let $h = d^{-1}\sum_{i=1}^d h_i$. Then h is weakly wandering and $S(h) = \bigvee_{i=1}^d S(h_i) = \bigvee_{i=1}^d(\mathbb{I} - E_i)$, since by Theorem 1(iii) one has $S(h_i) = \mathbb{I} - E_i$. ∎

THEOREM 3.2 (Stochastic multiparametric ergodic theorem): *Let α_i be automorphisms of von Neumann algebra M with finite normal state τ, $i = 1, 2, \ldots, d$. Then for $X \in L_1(M, \tau)$ averages $A_{*n}X$ converge stochastically to $\bar{X} \in L_1(M, \tau)$, where $n = (n_1, n_2, \ldots, n_d)$. Limit $\bar{X}$ is α_{*i} invariant and $\tilde{E}\bar{X}\tilde{E} = 0$, where $\tilde{E} = \bigvee_{i=1}^{d}(\mathbb{I} - E_i)$, and E_i projectors were constructed by Theorem 3.1.*

The proof of the theorem is based on the following theorem.

THEOREM 3.3 (see [17], [12], [5]): *Let M be a von Neumann algebra with normal tracial state τ and α_i be automorphisms of algebra M, $i = 1, \ldots, d$; ρ is a faithful normal α_i-invariant ($i = 1, \ldots, d$) state on M. Then for every $\mu \in M_*$ there exists an α_i-invariant functional $\bar{\mu}$ such that for every $\varepsilon > 0$ there exists projector $E \in M$, $\tau(E) > 1 - \varepsilon$; moreover $||A_{*n}\mu - \bar{\mu}||_1 \to 0$ and*

$$\sup_{\substack{x \in EM_+E \\ x \neq 0}} |(A_{*n}\mu - \bar{\mu})(x)/\tau(x)| \to 0 \quad \text{when multiindex } n \to \infty.$$

Let P_i be a map $\nu \to \lim_{k\to\infty} A_{*i}^k \nu$. Map P_i is a projection on the set of α_{*i} stationary points and

$$\bar{\mu} = P_d \cdot P_{d-1} \cdots P_1\mu.$$

Proof: For $d = 1$ the statement of the theorem coincides with Theorem 1.2. The induction steps are based on the following estimates. Fix $\varepsilon > 0$. There exists multiindex $(N_{d-1}, \ldots, N_1)^{(j)}$ such that for $n > (N_{d-1}, \ldots, N_1)^{(j)}$

$$||A_{*n_d}(A_{*(n_{d-1},\ldots,n_1)}P_{d-1}P_{d-2}\ldots P_1\mu)||_1 < \gamma_j,$$

where γ_j are defined in Theorem 2.3. There exists projector $E^{(j)}$ such that

$$\tau(E^{(j)}) > 1 - \varepsilon \cdot 2^{-j}$$

and

$$\sup_{\substack{x \in E^{(j)} M_+ E^{(j)} \\ x \neq 0}} |A_{*n_d}(A_{*(n_{d-1},\ldots,n_1)}\mu - P_{d-1} \cdot \ldots P_1\mu)(x)|/\tau(x) < 2^{-j}.$$

By Theorem 2.3 there exist N_d and projector $E^{(0)}$ such that $\tau(E^{(0)}) > 1 - \varepsilon/2$, and for $n > N_d$

$$\sup_{\substack{x \in E^{(0)} M_+ E^{(0)} \\ x \neq 0}} |A_{*n_d}(P_{d-1} \cdots P_1\mu) - P_d P_{d-1} \cdots P_1\mu)(x)|/\tau(x) < \delta/2.$$

Let $E = \bigwedge_{j=0}^{\infty} E^j$. Then $\tau(E) \geq 1 - \sum_{i=0}^{\infty} \tau(E^{(j)}) \geq 1 - \varepsilon$ and

$$\sup_{x \in EM_+E} |(A_{*(n_d \ldots n_1)}\mu - P_d \cdots P_1\mu)(x)/\tau(x)|$$

$$\leq \sup_{\substack{x \in EM_+E \\ x \neq 0}} |(A_{*n_d}(A_{*(n_{d-1} \ldots n_1)}\mu - P_{d-1} \cdots P_1\mu)(x)|/\tau(x)$$

$$+ \sup_{\substack{x \in EM_+E \\ x \neq 0}} |(A_{*n_d}(P_{d-1} \cdots P_1\mu) - P_d \cdots P_1\mu)(x)|/\tau(x)$$

$$< \delta,$$

when $j > |\log_2 \delta^{-1}/2| + 1$ and $n > (N_d, (N_{d-1}, \ldots, N_1)^{(j)})$. ∎

The proof of Theorem 3.2 is similar to the proof of Theorem 2.1.

4. Superadditive stochastic ergodic theorem

Let $\mathcal{F}$ be a set of all non empty intervals $[u, v[$; $u, v \in \mathbb{V}$. For $I \in \mathcal{F}$ and $w \in \mathbb{V}$ let $I + w = [u + w, v + w[$.

Definition 4.3: (see [10], [1]) The superadditive process with respect to automorphisms $\alpha_1, \alpha_2, \ldots, \alpha_d$ is called function F: $\mathcal{F} \ni I \to F_I \in M_{*h}$, where M_{*h} is the set of all hermitian normal functionals on von Neumann algebra M with the following properties:

(i) $F_i \circ \alpha^{(n)} = F_{I+n}$ for $I \in \mathcal{F}$, $n \in \mathbb{V}$; $\alpha^n = \alpha_1^{n_1} \ldots \alpha_d^{n_d}$;

(ii) for non-intersecting sets $I_1, I_2, I_3 = I_1 \cup I_2; I_3 \in \mathcal{F}$, we have

$$F_{I_3} \geq F_{I_1} + F_{I_2};$$

(iii) constant $\gamma(F) = \sup((\operatorname{card}(I))^{-1} F_I(\mathbb{I})) < \infty$.

The following theorem is valid:

THEOREM 4.1 (see [10], [1]): *Let M be a von Neumann algebra with faithful normal tracial state τ. Let also $\alpha_1, \ldots, \alpha_d$ be automorphisms of algebra M and F_I be a superadditive process with respect to $\alpha_1, \alpha_2, \ldots, \alpha_d$. Then $\bar{F}_{[0,n[} = (\operatorname{card}([0, n[))^{-1} F_{[0,n[}$ converge stochasticaly to α_{i*} invariant functional $\tilde{F}_0$, when multiindex $n \to \infty$.*

The proof of Theorem 4.1 is based on the following statements.

THEOREM 4.2 (see [17], [5]): *Let M be a von Neumann algebra with faithful normal tracial state τ and let $\alpha_1, \dots, \alpha_d$, $F_I, \tilde{F}_0$ be the same as above. Assume that there exists a faithful normal state which is α_{i*}-invariant ($i = 1, \dots, d$). Then $F_{[0,n[}$ converges to $\tilde{F}_0$ in the norm of the space M_*.*

The finite subset $\{I_\ell\}_{\ell=1}^k \subset \mathcal{F}$ is called admissible if there exists a rearrangement P of $\{1, \dots, k\}$ such that each $((\dots(I_{P(1)} \cup I_{P(2)}) \cup I_{P(3)}) \cup \dots \cup I_{P(\ell)})$ is in $\mathcal{F}$. Note that for an admissible subset $\{I_\ell\}_{\ell=1}^k$ property (ii) implies

$$F_{\bigcup\limits_{\ell=1}^{k} I_\ell} \geq \sum_{\ell=1}^{k} F_{I_\ell}.$$

Finite subset $\{I_{[m(j-1),mj[}\}$, $j \in [0,n[$, where $m \in \mathbb{V}$ and $mj = (m_1 j_1, \dots, m_d j_d)$; $m = (m_1, \dots, m_d)$, $j = (j_1, \dots, j_d)$, is admissible. For interval $I_{[0,n[}$ we can find a finite subset $\{I_\ell\}_{\ell=2}^k$ such that $I_{[0,n[} \cup \bigcup_{\ell=2}^k I_\ell = I_{[0,m[}$ and the set $\{I_{[0,n[}, \{I_\ell\}_{\ell=2}^k\}$ is admissible. Let $n!$ be a multiindex $(n!, \dots, n!)$ and $\bar{\bar{F}}_I$ be a limit in the norm M_* of $A'_n(\bar{F}_I)$ when $n \to \infty$. Let us show that $\bar{\bar{F}}_{[0,n![}$ increases. Indeed, from (ii) it follows that

$$F_{[0,n![} \geq \sum_{\substack{0 \leq i_j < n \\ j=1,\dots,d}} \alpha_1'^{(n-1)! \cdot i_1} \dots \alpha_d'^{(n-1)! \cdot i_d} F_{[0,(n-1)![}. \tag{7}$$

Applying A_k to both sides of (7) and tending to the limit in norm M_* when $k \to \infty$, one gets

$$\tilde{F}_{[0,n![} \geq \sum_{\substack{0 \leq i_j < n \\ j=1,\dots,d}} \tilde{F}_{[0,(n-1)![}.$$

Let us show now that if F is a non-negative superadditive process (or $F_I \geq 0$ for all $I \in \mathcal{F}$), then $\lim_{n\to\infty} \bar{F}_{[0,n[}(\mathbb{I}) = \gamma(F)$. Indeed, fix $\varepsilon > 0, k$ such that $\bar{F}_{[0,k[}(\mathbb{I}) > \gamma(F) - \varepsilon/2$. Let n be such that

$$(\mathrm{card}([0,n[))^{-1}\mathrm{card}([0,[m/k]k[) > \gamma',$$

where $\gamma' = 1 - \varepsilon/2(\max\{1,\gamma\})^{-1}$; $[n/k] = ([n/k_1], [n/k_2], \dots, [n/k_d])$. Then

$$\begin{aligned}
\bar{F}_{[0,n[}(\mathbb{I}) &\geq \mathrm{card}([0,n[))^{-1} F_{[0,[n/k]k[}(\mathbb{I}) \\
&\geq \mathrm{card}([0,n[))^{-1}\mathrm{card}([0,[n/k]k[) \cdot \bar{F}_{[0,[n/k]k[}(\mathbb{I}) \\
&\geq (1-\varepsilon/2)(\gamma - \varepsilon/2) > \gamma - \varepsilon;
\end{aligned}$$

here every inequality is valid because all the indexes are admissible.

We now show that Theorem 4.1 is enough for the proof for a non-negative superadditive process. Indeed,

$$\mathrm{card}([0,k[))^{-1}\sum_{i\in[0,k[}\alpha_1^{i_1}\dots\alpha_d^{i_d}F_{(0,\dots,0)}\to\tilde{F}_{(0,\dots,0)}$$

stochastically. Consider $F'_I = F_I - \sum_{i\in I}\alpha_1'^{i_1}\dots\alpha_d'^{i_d}F_{(0,\dots,0)}$. Then $F'_{(0,\dots,0)} = 0$ and $F'_I \geq \sum_{i\in I}\alpha_1'^{i_1}\dots\alpha_d'^{i_d}F'_{(0,\dots,0)} = 0$. Note also that stochastical convergence of $\bar{F}'_I$ is equivalent to stochastical convergence of $\bar{F}_I$. Assume that F_I is a non-negative superadditive process and $\varepsilon > 0$. Fix $n \in \mathbb{V}$ such that $\tilde{\bar{F}}_{[0,n![}(\mathbb{I}) > \gamma(F)-\varepsilon$. Let $\tilde{F}_0 = \uparrow - \lim_{k\to\infty}\tilde{\bar{F}}_{[0,k![}$. Then

$$\tilde{F}_0(\mathbb{I}) = \gamma(F) \quad \text{and} \quad ||\tilde{\bar{F}}_{[0,n![}-\tilde{F}_0|| = \tilde{F}_0(\mathbb{I}) - \tilde{F}_{[0,n![}(\mathbb{I}) < \varepsilon.$$

Let $k_0 \in \mathbb{V}$ such that for $k > k_0$ and $k \in \mathbb{V}$,

$$(\mathrm{card}([0,k[))^{-1}\mathrm{card}([i,([k/n!]-1)\cdot n!+i[) > 1-\varepsilon \qquad \text{for all } i\in[0,n![$$

holds and $\bar{F}_{[0,k[}(\mathbb{I}) \geq \gamma - \varepsilon$. Let k_1 be such that for $k > k_1$,

$$||A_{([k/n!])n!}(\bar{F}_{[0,n![}) - \tilde{\bar{F}}_{[0,n![}||_1 \leq \varepsilon$$

holds. Then for $k > \max\{k_0,k_1\}$ the following inequality is valid:

(8)
$$\begin{aligned}
&||\bar{F}_{[0,k[}-\tilde{F}_0||_1\\
\leq&(\mathrm{card}([0,n![))^{-1}\cdot\sum_{i\in[0,n![}||\bar{F}_{[0,k[}-(\mathrm{card}([0,[k.n!]-1[))^{-1}\\
&\sum_{j\in[0,[k/n!]-1[}\alpha_1'^{n!\cdot j_1}\cdot\dots\cdot\alpha_d'^{n!\cdot j_d}(\alpha'^{i}F_{[0,n![})||_1+||A_{([k/n![)n!}\bar{F}_{[0,n![}-\tilde{\bar{F}}_{[0,n![}||_1\\
&+||\tilde{\bar{F}}_{[0,n![}-\tilde{F}_0||_1.
\end{aligned}$$

Let us estimate each term on the right side of (8). By the choice of n the last term does not exceed ε, and by the choice of k the third term does not exceed ε.

For the first term we have:

$$\begin{aligned}
(\text{F.T.}) \leq & (\text{card}([0,k[))^{-1} \| F_{[0,k[} - \sum_{j\in[0,[k/n!]-1[} \alpha_1^{\prime n! j_1} \dots \alpha_d^{\prime n! j_d} (\alpha^{\prime i} F_{[0,n![}) \| \\
& + (1 - (\text{card}([0,k[))^{-1} \cdot (\text{card}([0,n![)) \\
& \qquad \cdot (\text{card}([0,[k/n!]-1[))) \cdot \| F_{[0,n![} \| \\
\leq & (\text{card}([0,k[))^{-1} [F_{[0,k[}(\mathbb{I}) - \text{card}([0,[k/n!]-1[) \cdot F_{[0,n![}(\mathbb{I})] \\
& + (1 - \text{card}([0,k[))^{-1} \cdot \text{card}([0,n![)\text{card}([0,[k/n!]-1[) \cdot F_{[0,n[}(\mathbb{I}) \\
\leq & (\text{card}([0,k[))^{-1} \\
& \cdot [\text{card}([0,k[) \cdot (\gamma - \varepsilon) - \text{card}([0,[k/n!]-1[)\text{card}([0,n![) \cdot \gamma] \\
& + (\text{card}([0,k[))^{-1} [\text{card}([0,k[) - \text{card}([0,[k/n!]-1[)) \cdot \text{card}([0,n![)]) \cdot \gamma \\
\leq & 4\varepsilon\gamma,
\end{aligned}$$

where the second inequality follows from the possibility of admissible extending of $\{F_{[jn!+i,(j+1)n!+i[}\}$ where $j \in [0,[k/n!]-1[$ up to set $[0,k[$ and from the non-negativity of the superadditive process; the last inequality follows from the choice of k. Thus (8) does not exceed $2\varepsilon + 4\varepsilon\gamma$. ∎

Proof of Theorem 4.1: Similar to the proof of Theorem 3.1.

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